Math 131B-1: Homework 6

Due: February 19, 2014

1. Read Apostol Sections 9.9-10, 9.12, 9.18-19, 8.24, 9.22. [It may also be helpful to read 9.14-15, but note that these sections are phrased in terms of complex power series. We'll say a little bit about complex numbers next week.]

2. Do Apostol problems 9.31, 9.33, 9.35. [Problem 9.35 is much easier if you prove the result of 9.13.]

3. Recall that $B(S \to T)$ (or any of the other notations mentioned in class) is the set of bounded functions $f\colon S \to T$ and $C(S \to T) \subset B(S \to T)$ is the subspace consisting of bounded continuous functions.

   - Prove that the operation $d_\infty(f, g) = \sup_{x \in S} d_T(f(x), g(x))$ is a metric on $B(S \to T)$.
   - Give an example showing that if T is not complete, $C(S \to T)$ need not be complete.

4. Find the Taylor series expansions for $e^x \sin x$ about $\frac{\pi}{2}$ and $\ln(1+x)\tan^{-1}(x)$ about 0. Note that if your solution involves the product rule, you're working much too hard.

5. Prove that $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ if $a_{ij} \geq 0$ for all i and j, as long as we allow the case $+\infty = +\infty$ to occur.

6. *A question of arc length.* Recall that the sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges uniformly to $f(x) = 0$ on the real line. Moreover, recall, e.g. from your calculus class, that whenever g is a continuous function on $[a, b]$ which is differentiable on (a, b) with continuous derivative g', the arclength of the curve $\{(x, g(x)) \in \mathbb{R}^2 : x \in [a, b]\}$ is $S_a^b(f) = \int_a^b \sqrt{1 + f'(x)^2}dx$. Show that $S_a^b(f_n)$ does not converge to $S_a^b(f)$. [Hint: You can't actually do the integrals you get, you're looking for a lower bound which is greater than π.]

   Ergo it is possible for a sequence of functions to converge uniformly on an interval without convergence of the arc lengths of their graphs over the interval. What kind of hypothesis do you think you would need to add to get convergent arc lengths?

7. *Continuity makes life easier.* In class we proved that if $f_n\colon (a, b) \to \mathbb{R}$ is a sequence of differentiable functions such that the derivatives f_n' converge uniformly to some g on (a, b) and $\{f_n(x_0)\}$ converges for at least one x_0 in (a, b), then there is a differentiable function f such that $f_n \to f$ uniformly, and $f'(x) = g(x)$. If we're willing to add the assumption that each of the f_n' is continuous on (a, b), we can give an easier proof.

   - Show that $\int_{x_0}^x f_n'$ and $\int_{x_0}^x g$ both exist for every $x \in (a, b)$, and the sequence of functions $\int_{x_0}^x f_n'$ converges uniformly to $\int_{x_0}^x g$. [Hint: This is extremely straightforward.]
   - Observe that by FTC Part I, $\int_{x_0}^x f_n' = f_n(x) - f_n(x_0)$. Therefore $h_n(x) = f_n(x) - f_n(x_0)$ converges uniformly to $\int_{x_0}^x g$.
   - Let $L = \lim_{n\to\infty} f_n(x_0)$. Consider the function $f\colon (a, b) \to \mathbb{R}$ defined by $f(x) = L + \int_{x_0}^x g$. Prove, using the second part of this problem, that $f_n(x) \to f(x)$ uniformly.
   - Prove, using FTC Part II, that $f'(x) = g(x)$ on (a, b). [This is the only place where it is important that the f_n' are continuous, and not merely integrable.]